Journal of Statistical Physics, Vol. 2, No. 4, 1970

On the Detectabillity Limit of Coherent Optical Signals in Thermal Radiation

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Received August 27, 1970

The problem of detecting a completely known coherent optical signal in a thermal background radiation is considered. The problem is a quantum mechanical analog of detection of a known signal in Gaussian noise. The quantum detection counterpart is formulated in terms of a pair of density operators and a solution is shown to exist. A perturbation solution is obtained by making use of a reproducing kernel Hilbert space of entire functions. The solution is particularly applicable to optical frequencies, where the effect of thermal radiation is small, and it is shown to converge to known results at zero thermal radiation. Curves are generated showing the detectability limit at optical frequencies. Also considered is the problem of finding an operator that maximizes a signal-to-noise ratio, defined for quantum detection in analogy with the classical theory. For a coherent signal with random phase, the operator that maximizes the signal-to-noise ratio is identicial to the one obtained by applying the Neyman–Pearson criterion, thereby establishing a complete analogy with the classical detection theory. For a signal with known phase, however, the analogy breaks down in the limit of zero thermal radiation. In that case, it is shown that an operator that maximizes the "classical" signal-to-noise ratio does not exist.

KEY WORDS: Signal detection; detection theory; quantum statistics; optical signal; thermal radiation; coherent optics; signal-to-noise ratio.

1. INTRODUCTION

The possibility of using lasers in interplanetary communication systems has stimulated recent research in the detection and estimation theory of signals at optical frequencies. In particular, Helstrom's work[1–4] laid the foundation for the development of a rigorous communication theory of optical signals.

A part of this paper has been submitted to the University of California, Los Angeles, in partial satisfaction of the Ph.D. requirement.

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A new theory of signal detection is needed at optical frequencies in the same way in which quantum mechanics was needed to complement classical mechanics. The conventional signal detection theory of frequencies up to microwaves corresponds to classical mechanics in that the theory could ignore the effects of the uncertainty principle. At optical frequencies, as in quantum mechanics, the uncertainty principle imposes a limitation on the precision achievable in a simultaneous measurement. This lack of precision may be incorporated as receiver noise. For example, the noise power density of an ideal linear amplitude or phase detector is[5]

$$S(f) = hf/(e^{hf/kT} - 1)$$

where h is Planck's constant, k is Boltzmann's constant, T is absolute temperature, and f is the frequency. The first factor is thermal noise and the second factor is quantum in origin. At small f such that $hf \ll kT$, the thermal noise dominates. At high frequencies such that $hf \gg kT$, the "quantum" noise dominates over the thermal noise. At ordinary temperatures, the frequency that separates "quantum" region from "thermal" region occurs at 10^{13} Hz, in the infrared specturm. Thus, if a detection theory is to be successful at optical frequencies, it is clear that the theory must take into account the quantum nature of signal and noise.

In the following, we consider the detection problem in quantum mechanics as posed by Helstrom.[2] While it is not the only consistent way to formulate the problem, it is appealing in its simplicity and similarity to the classical formalism. In Section 2, we formulate the general quantum detection problem with some care and show that a solution exists. Within this formalism, we wish to consider the problem of detecting a coherent, completely known signal in background thermal radiation. This, according to Helstrom,[6] is "an outstanding unsolved problem of quantum detection theory." We are interested in this problem because its solution gives the upper limit of our ability to communicate using a coherent optical source such as the laser.

In Section 3, we introduce some of the background material for the problem and indicate where the difficulty of the problem lies. We also show how the Hilbert space of entire functions follows naturally from a consideration of a coherent signal and show some properties of the space, among them, the all-important reproducing property.

By repeatedly using the reproducing property, we solve the problem in steps in Section 4. First, we solve the case of zero thermal radiation and show that the solution is exactly the one obtained earlier by Helstrom by a different approach. We then generalize this solution to derive an algorithm that is capable of complete solution. We present some numerical results applicable to optical frequencies, where the contribution due to thermal radiation is small.

In Section 5, we consider the problem of finding an operator that maximizes a signal-to-noise ratio, which Helstrom defined by analogy to the classical theory.[2] For a coherent signal of random phase, we find that the operator that maximizes the signal-to-noise ratio is identical to the one obtained by applying the Neyman–Pearson criterion. We find a corresponding operator for a coherent known signal of small amplitude. In the limit of zero thermal radiation, we show that an operator that maximizes the signal-to-noise ratio does not exist.

2. BINARY DETECTION IN QUANTUM STATISTICS

The classical detection theory can be considered an application of the hypothesis testing formalism[7,8] Helstrom[2] showed that the quantum detection problem can also be posed as one of hypothesis testing. To see the parallel between the two, the classical theory is reviewed briefly.

The detection of a known signal in noise is a binary hypothesis testing problem in that we observe some output X of the receiver and decide either: H_1, signal is present; or H_0, the output contains noise only. The value X attains on measurement is determined by the conditional distribution $F_\theta(X)$, which depends on a real parameter θ. In the following, we consider the two hypotheses to be simple, i.e., the distribution of X under hypothesis H_i is determined completely by $F_{\theta=\theta_i}(X)$, where the parameter θ takes on the value θ_i.

The classical hypothesis testing procedure directs us to find a best decision function $\phi(X)$.[9] The domain of ϕ is the sample space and its range is the [0, 1] interval. The interpretation of $\phi(X)$ is that, if $X = x$ is observed, we choose H_1 with probability $\phi(X)$. It follows that we must choose H_0 with probability $1 - \phi(X)$. The probability of deciding a signal is present when actually there is none is the probability of false alarm, given by

$$Q_0 = E_{\theta=\theta_0}\phi(X) \equiv E_0\phi(X) = \int \phi(x)\, dF_0(x)$$

The probability of correctly deciding that a signal is present is

$$Q_1 = E_{\theta=\theta_1}\phi(X) \equiv E_1\phi(X) = \int \phi(x)\, dF_1(x)$$

Which decision rule is "best" obviously depends on the criterion used. The two most common are the Bayes criterion, which seeks to minimize the average cost of making a wrong decision, and the Neyman–Pearson criterion, which maximizes Q_d for a given Q_0. To apply the Bayes criterion, the prior probabilities of H_0 and H_1 and the relative cost of making a wrong decision must be known. It can be shown (Helstrom,[2] p. 261) that the detection problem reduces to one of maximizing the functional

$$E_1\phi(X) - \lambda E_0\phi(X) \tag{1}$$

by choosing ϕ that satisfies the constraint $0 \leqslant \phi(X) \leqslant 1$.

If the Neyman–Pearson criterion is used, we fix the false alarm probability $E_0\phi(X)$ at some level α_1 and try to find a function ϕ which maximizes the detection probability $E_1\phi(X)$. Introducing the Lagrangian multiplier λ, we again maximize $E_1\phi(X) - \lambda E_0\phi(X)$ subject to the same constraint as above.

The problem is further developed by assuming some distribution of random variable X, but for our purposes we need not take the problem any further. It is sufficient to note at this point that the classical detection problem may be formulated as one of maximizing a linear functional (1) subject to a positivity constraint.

In the discovery of quantum mechanics, the classical mechanics served as a guide in developing new theories. In the same way, we look to the classical signal detection theory to guide us in building a quantum statistical theory of signal detection. Helstrom[3] constructed such a theory by closely adhering to the formalism of classical hypothesis testing. We shall now reconstruct his theory and show the consistency of his formulation.

In quantum mechanics, the set of variables characterizing a given physical system is put in a one-to-one correspondence with a set of self-adjoint operators. Defining the state of a physical system is equivalent to measuring all the variables corresponding to the operators in a certain compatible set. This set is compatible in that all the operators commute with each other and the set is complete in the sense that there exist no other operator that commutes with each in the set.

The space on which these operators act is assumed to be a separable Hilbert space, H. The elements of H are vectors $|\, x\rangle$ and H is complete in the metric generated by the inner product $\langle x \,|\, y\rangle$. The space H is defined in such a way that all operators representing physical observables have eigenvectors that span H.[10]

If an operator A corresponds to an observable and the system is in some known state $|\, e\rangle$, the average value of A is given by $\langle A\rangle = \langle e \,|\, A \,|\, e\rangle$. Often, the state of the system is incompletely known and admits only a statistical description. For example, we may only know that the system is in state $|\, e_k\rangle$ with probability p_k. In that case, the average of A is given by the trace of A, i.e.,

$$\begin{aligned}\langle A\rangle &= \sum_k p_k \langle e_k \,|\, A \,|\, e_k\rangle = \operatorname{Tr} \sum_k p_k \,|\, e_k\rangle\langle e_k \,|\, A \\ &= \operatorname{Tr} \rho A\end{aligned}$$

where ρ is a density operator given by $\sum_k p_k \,|\, e_k\rangle\langle e_k \,|$. Such a representation of ρ in terms of a complete orthonormal set $\{|\, e_k\rangle\}$ is possible because H is separable.[11] Moreover, $\operatorname{Tr} \rho A$ thus defined is independent of the basis $|\, e_k\rangle$.[11,13] We list some properties of a density operator. First, a density operator is bounded. For any $|\, f\rangle$ in H,

$$\|\rho \,|\, f\rangle\|^2 \leqslant \sum_k p_k \,|\langle f \,|\, e_k\rangle|^2 \leqslant \sum_k |\langle f \,|\, e_k\rangle|^2 = \|\,|\, f\rangle\|^2$$

Second, it is positive since

$$\langle f \,|\, \rho \,|\, f\rangle = \sum_k p_k \,|\langle f \,|\, e_k\rangle|^2 \geqslant 0$$

Third, its trace is unity,

$$\operatorname{Tr} \rho = \sum_k p_k \langle e_k \,|\, e_k\rangle = \sum_k p_k = 1$$

The positivity of ρ and the fact that it has unity trace confirm our suspicion that operator $\operatorname{Tr} \rho_\theta(\cdot)$ is analogous to the expectation $E_\theta(\cdot)$ of the classical statistics. We rely on this analogy to build a quantum statistical theory of signal detection.

To solve the classical hypothesis testing problem, it was sufficient to find a best decision function $\phi(X)$, a mapping from the sample space to the [0, 1] interval. In quantum statistical hypothesis testing, the sample space is the set of complete compatible operators mentioned earlier. In analogy with classical hypothesis testing, we seek a mapping from the set of compatible operators to [0, 1]. This mapping must also be an observable just as $\phi(X)$ was a random variable. We consider, in place of $\phi(X)$, an observable represented by a self-adjoint operator π, which is a function of the compatible operators such that all of its eigenvalues are in [0, 1]. Then, the probability of false alarm is

$$Q_0 = \text{Tr}\, \rho_{\theta=\theta_0}\pi = \text{Tr}\, \rho_0\pi$$

and the probability of detection is

$$Q_1 = \text{Tr}\, \rho_{\theta=\theta_1}\pi = \text{Tr}\, \rho_1\pi$$

Since π represents an observable, it has a complete orthonormal set of eigenvectors $| u_k\rangle$ corresponding to eigenvalues μ_k. Then, for any $|f\rangle$ in H of norm 1, we have $|f\rangle = \sum_k f_k \,| u_k\rangle$, where $\sum_k |f_k|^2 = 1$, and since $0 \leqslant \mu_k \leqslant 1$ for all k,

$$0 \leqslant \langle f|\,\pi\,|f\rangle = \sum \mu_k\,|f_k|^2 \leqslant 1$$

Carrying the analogy with classical hypothesis testing a step further, we say that the quantum mechanical counterpart is to find a self-adjoint operator π such that

$$\text{Tr}\, \rho_1\pi - \lambda\, \text{Tr}\, \rho_0\pi \tag{2}$$

is maximized subjected to the constraint that

$$0 \leqslant \langle f|\,\pi\,|f\rangle \leqslant 1 \tag{3}$$

for any $|f\rangle$ of norm 1.

The problem is still much too general to be handled, and to make further progress, we appeal to the theory of trace-class operators in H. Trace-class or nuclear operators are a subclass of Hilbert–Schmidt operators having finite trace. Density operators, by virtue of their unity trace, belong to it. The trace-class operators form a normed linear space over the reals and a product of a trace-class operator and a bounded operator is again trace-class.[12]

Since density operators ρ_1 and ρ_0 are trace-class and π is a bounded operator, $\rho_1\pi$ and $\rho_0\pi$ are trace-class. Hence, so is $(\rho_1 - \lambda\rho_0)\,\pi$, λ being a positive real parameter. It follows that the evaluation of $\text{Tr}(\rho_1 - \lambda\rho_0)\,\pi$ is independent of the basis used. We chose to compute $\text{Tr}(\rho_1 - \lambda\rho_0)\,\pi$ on the basis composed of the eigenvectors of $\rho_1 - \lambda\rho_0$. Such a basis exists since $\rho_1 - \lambda\rho_0$, being a trace-class operator, is *a fortiori* compact, and a compact self-adjoint operator has a complete orthonormal set of eigenvectors (Akhiezer and Glazman, p. 131).[11]

Accordingly, we let $\rho_1 - \lambda\rho_0$ satisfy the eigenvalue equation

$$(\rho_1 - \lambda\rho_0)\,|\,\eta_k\rangle = \eta_k\,|\,\eta_k\rangle \tag{4}$$

where η_k is the kth eigenvalue and $|\,\eta_k\rangle$ is the corresponding eigenvector. Then,

$$\text{Tr}(\rho_1 - \lambda\rho_0)\,\pi = \sum_k \langle\eta_k\,|\,(\rho_1 - \lambda\rho_0)\,\pi\,|\,\eta_k\rangle = \sum_k \eta_k\langle\eta_k\,|\,\pi\,|\,\eta_k\rangle \tag{5}$$

The detection problem is to find a self-adjoint operator π that maximizes (5) subject to the constraint (3).

We claim π_0 given by

$$\pi_0 = \sum_{k:\eta_k \geqslant 0} | \eta_k\rangle\langle\eta_k | \tag{6}$$

is one such operator. After Helstrom, we call π_0 the detection operator. That π_0 satisfies (3) is immediate. To show that π_0 maximizes (5), we consider any other operator π and show that $\text{Tr}(\rho_1 - \lambda\rho_0)(\pi_0 - \pi)$ is nonnegative. Since eigenvectors $| \eta_k\rangle$ are complete, the most general form of a bounded operator π is

$$\pi = \sum_{k,l} | \eta_k\rangle\langle\eta_k | \pi | \eta_l\rangle\langle\eta_l | = \sum_{k,l} \pi_{kl} | \eta_k\rangle\langle\eta_l |$$

where $\pi_{kl} = \langle\eta_k | \pi | \eta_l\rangle$. Then,

$$\begin{aligned}\text{Tr}(\rho_1 - \lambda\rho_0)(\pi_0 - \pi) &= \sum_{k:\eta_k \geqslant 0} \langle\eta_k | (\rho_1 - \lambda\rho_0) | \eta_k\rangle - \sum_{k,l} \pi_{kl} | \rho\rho_1 - \lambda\rho_0 | \eta_k\rangle \\ &= \sum_{k:\eta_k \geqslant 0} \eta_k - \sum_{k,l} \pi_{kl}\eta_k\delta_{kl} \\ &= \sum_{k:\eta_k \geqslant 0} (1 - \pi_{kk})\,\eta_k - \sum_{k:\eta_k < 0} \eta_k\pi_{kk} \geqslant 0\end{aligned}$$

where the last inequality results because

$$0 \leqslant \pi_{kk} = \langle\eta_k | \pi | \eta_k\rangle \leqslant 1$$

If an operator π_a is found such that

$$\text{Tr}(\rho_1 - \lambda\rho_0)(\pi_0 - \pi_a) = 0 \tag{7}$$

we cannot conclude that $\pi_0 = \pi_a$, because $\rho_1 - \lambda\rho_0$ is not positive. Thus, the detection operator π_0 given by (6) is not unique in maximizing $\text{Tr}(\rho_1 - \lambda\rho_0)\,\pi$. However, all operators π_a satisfying (7) are equivalent to π_0 in the sense of achieving the same performance as π_0.

From its definition of (6), the detection operator π_0 is seen to be a projection operator onto a subspace spanned by the eigenvectors corresponding to the positive eigenvalues of $\rho_1 - \lambda\rho_0$. Being a projection operator, π_0 has only two distinct eigenvalues: either 0 or 1. This means a measurement of the observable represented by π_0 gives either 0 or 1. If the measured values is 1, we decide H_1 is true with probability 1; otherwise, we decide H_0 is true. Thus, π_0 is analogous to the nonrandomized decision rule of classical hypothesis testing.

There remains the problem of finding the detection operator π_0 for a given pair of density operators ρ_1 and ρ_0. To obtain π_0, we must solve the eigenvalue equation (4) and construct π_0 as in (6). If it happens that ρ_1 and ρ_0 commute, then there exists a basis composed of simultaneous eigenvectors of ρ_1 and ρ_0. If the eigenvalue solution is known for either ρ_1 or ρ_0, the construction of π_0 is trivial. But if ρ_1 or ρ_0 does not commute, and the eigenvalue equation (4) cannot be solved easily, then the problem is very difficult.

One possible approach is the simultaneous diagonalization of ρ_1 and ρ_0. Being linear, nonnegative, and trace-class, density operators are also covariance operators. Although some necessary and sufficient conditions for simultaneous diagonalization of two such operators in Hilbert space are known,[14] for the present problem the usefulness of the method appears limited.

An interesting special case of the above problem occurs in the detection of a coherent, completely known signal in thermal radiation. For this problem, ρ_1 and ρ_0 do not commute and the detection operator π_0 has not been found. In the next section, we discuss the background of the particular problem and introduce a representation of ρ_1 and ρ_0 convenient for later development.

3. DENSITY OPERATORS IN COHERENT REPRESENTATION

Our aim is to find the detectability limit of coherent optical signal in thermal radiation. Accordingly, we consider an idealized communication system where all system imperfections are removed.[2] In such a system, the source emits a perfectly coherent signal toward the receiver. The transmitting medium is vacuum and the receiver is a cavity with an aperture through which the signal together with background thermal radiation is introduced. The cavity is initially empty and it is exposed to the incoming radiation for a time that is much longer than the period of any signal oscillations. After the aperture is closed, we measure the field inside the cavity, as precisely as allowed by the uncertainty principle, to determine whether the field contained any signal component.

The electromagnetic field of signal and noise, within and exterior to the cavity, can be quantized in the usual way[15,16] and expressed in terms of the conjugate pair amplitudes a_k and a_k^+ for the kth mode, $1 \leqslant k < \infty$. In the detection problem, however, we need to consider only a finite number of modes since a coherent source can put energy in at most a finite number of modes. The modes not containing any signal component are irrelevant to the problem and can be ignored. Moreover, by introducing a suitable unitary transformation, we can further reduce the problem to considering just one mode (Helstrom,[3] p. 48).

Thus, the field of the appropriate mode may be given in terms of a pair of amplitude operators a, called the annihilation operator, and a^+, called the creation operator. The operators satisfy the commutation relations

$$\begin{aligned} [a, a^+] &= aa^+ - a^+a = I \\ [a, a] &= [a^+, a^+] = 0 \end{aligned} \tag{8}$$

The self-adjoint operator $\mathcal{N}$ defined by the product a^+a satisfies the eigenvalue equation[15,16]

$$\mathcal{N} \mid n\rangle = n \mid n\rangle \tag{9}$$

$\mathcal{N}$ is called the number operator because its eigenvalues are nonnegative integers, and $| n\rangle$ is the eigenvector corresponding to eigenvalue n. The number states $| n\rangle$ are orthonormal and complete in H.

As a consequence of the commutation relation (8), the annihilation operator a satisfies the eigenvalue equation[18,19]

$$a | \alpha\rangle = \alpha | \alpha\rangle \tag{10}$$

The eigenstate $| \alpha\rangle$ is called a coherent state and it corresponds to the state of a driven or displaced harmonic oscillator (Klauder and Sudarshan,[19] p. 106). The spectrum of a is all of the complex plane with multiplicity of one.

The coherent state $| \alpha\rangle$ can be expressed in terms of the number states $| n\rangle$ by[17,19,20]

$$| \alpha\rangle = [\exp(- | \alpha |^2/2)] \sum_{n=0}^{\infty} [\alpha^n/(n!)^{1/2}] | n\rangle \tag{11}$$

Coherent states $| \alpha\rangle$ are not orthogonal since

$$\begin{aligned}\langle\alpha | \beta\rangle &= [\exp(-\tfrac{1}{2} | \alpha |^2 - \tfrac{1}{2} | \beta |^2)] \sum_{n,m} [\alpha^{*n} \beta^m/(n!m!)^{1/2}]\langle n | m\rangle \\ &= \exp(\alpha^*\beta - \tfrac{1}{2} | \alpha |^2 - \tfrac{1}{2} | \beta |^2) \end{aligned}\tag{12}$$

but they are complete in the sense

$$I = \int | \alpha\rangle\langle\alpha | (d^2\alpha/\pi) \tag{13}$$

where $d^2\alpha = d(\text{Re}\, \alpha)\, d(\text{Im}\, \alpha)$ and the integration is over the entire complex plane. Equation (13) is formally established from (11), the completeness of $| n\rangle$, and the identity

$$\int \alpha^{*n} \alpha^m (\exp (- | \alpha |^2)(d^2\alpha/\pi) = n!\, \delta_{nm}$$

A rigorous proof of (13), justifying the interchange of summation and integration, is given by Bargmann.[20] Using (13), any $|f\rangle$ in H can be given the coherent representation

$$|f\rangle = \int \langle\alpha\rangle\langle\alpha | f\rangle(d^2\alpha/\pi) = \int | \alpha\rangle f(\alpha^*)[\exp(- | \alpha |^2/2)](d^2\alpha/\pi) \tag{14}$$

where $\langle\alpha | f\rangle = f(\alpha^*) \exp(-| \alpha |^2/2)$. Also, by the completeness of $| n\rangle$, $|f\rangle = \sum_n f_n | n\rangle$, where $f_n = \langle n | f\rangle$. Then, using (11),

$$\langle\alpha | f\rangle = [\exp(- | \alpha |^2/2)] \sum_{n=0}^{\infty} [f_n\alpha^{*n}/(n!)^{1/2}]$$

Equating the two expressions of $\langle\alpha | f\rangle$, we conclude

$$f(\alpha^*) = \sum_{n=0}^{\infty} [f_n\alpha^{*n}/(n!)^{1/2}] \tag{15}$$

In view of the condition $\sum_n |f_n|^2 = 1$, the series (15) is absolutely convergent for α finite, thus defining an entire function.

The class $\mathscr{F}$ of entire functions $f(\alpha^*)$ has some remarkable properties. Foremost is its reproducing property. If we multiply (14) by $\langle\beta|$,

$$\langle\beta | f\rangle = f(\beta^*) \exp(-|\beta|^2/2) = \int \langle\beta | \alpha\rangle f(\alpha^*)[\exp(-|\alpha|^2/2)](d^2\alpha/\pi)$$

and use $\langle\beta | \alpha\rangle$ given in (12), we have the reproducing formula

$$f(\beta^*) = \int [\exp(\beta^*\alpha - |\alpha|^2)] f(\alpha^*)(d^2\alpha/\pi) \tag{16}$$

If $f(\alpha^*) = 0$ for all α^*, then $|f\rangle = 0$, and conversely. Therefore, any $|f\rangle$ in H can be put into a one-to-one correspondence with $f(\alpha^*)$ in $\mathscr{F}$. Moreover, by (16), we have

$$\langle f | g\rangle = \int [f(\alpha^*)]^* g(\alpha^*)(\exp - |\alpha|^2)(d^2\alpha/\pi)$$

The integral on the right has the correct properties of an inner product and $\mathscr{F}$ can be made into a Hilbert space. We shall denote the inner product in $\mathscr{F}$ by (,) and make use of $\mathscr{F}$ in Section 4.

We now review the derivation of density operators ρ_1 and ρ_0, corresponding to hypothesis H_0 and H_1.[16] To obtain ρ_0, we assume the electromagnetic field of thermal radiation has been coupled to the appropriate cavity mode and that the cavity is at a thermodynamic equilibrium at temperature T. Under these conditions, the density operator is one that maximizes the entropy

$$S = -k\rho \ln \rho \tag{17}$$

under the constraint

$$\mathrm{Tr}\, \rho = 1 \tag{18a}$$

and

$$\mathrm{Tr}\, \rho H = \langle E\rangle \tag{18b}$$

where H is the Hamiltonian given by

$$H = hf(a^+a + \tfrac{1}{2})$$

and E is the known average energy of the mode. The result of maximization is (Louisell,[16] p. 232)

$$\rho_0 = e^{-H/kT}/\mathrm{Tr}\, e^{-H/kT}$$

Substituting for H, we obtain

$$\rho_0 = (1 - e^{-w})\, e^{-wa^+a} = (1 - e^{-w}) \sum_{n=0}^{\infty} e^{-wn} |n\rangle\langle n| \tag{19}$$

where $w = hf/kT$ and the second equality follows because $\exp(-wa^+a) = \exp(-w\mathscr{N})$ is diagonal in the number representation.

To derive the density operator under hypothesis H_1, we assume a coherent, completely known signal to be superposed on the thermal radiation. We still maximize (17) subject to (18a) and (18b), but we have two additional constraints:

$$\langle p \rangle = \mathrm{Tr}\ \rho p$$

$$\langle q \rangle = \mathrm{Tr}\ \rho q$$

These two conditions reflect our knowledge about the known signal. The density operator under H_1 is (Louisell,[16] p. 245)

$$\rho_1 = [(1 - \exp(-w)] \exp[-w(a^+ - \mu^*)(a - \mu)] \tag{20}$$

where $|\mu|^2 = N_s$, the average number of signal photons.

We now give the density operators ρ_0 and ρ_1 in the coherent state representation. By the completeness relation (13), density operator ρ_i has the representation

$$\rho_i = \iint |\alpha\rangle\langle\alpha| \rho_i |\beta\rangle\langle\beta| (d^2\beta/\pi)(d^2\alpha/\pi) \tag{21}$$

To compute the quantity $\langle\alpha | \rho_i | \beta\rangle$, we make use of the following operator identity. If an operator b and its adjoint b^+ satisfy the commutation relations

$$[b, b] = [b^+, b^+] = 0, \qquad [b, b^+] = I$$

then for a real parameter χ (Dirac,[15] p. 116),

$$\exp(\chi b^+ b) = \sum_{m=0}^{\infty} [(e^{\chi} - 1)^m/m!]\, b^{+m} b^m$$

We apply this formula to evaluate $\langle\alpha | \rho_1 | \beta\rangle$, by letting $b = a - \mu$. Then, $[b, b^+] = [a, a^+] = 1$, so that

$$\begin{aligned}\langle\alpha | \rho_1 | \beta\rangle &= (1 - e^{-w}) \sum_{m=0}^{\infty} [(e^{-w} - 1)/m!]\langle\alpha | (a^+ - \mu^*)^m (a - \mu)^m | \beta\rangle \\ &= (1 - e^{-w}) \exp[(e^{w} - 1)(\alpha^* - \mu^*)(\beta - \mu)]\langle\alpha | \beta\rangle \\ &= (1 - v_0) \exp[v_0\alpha^*\beta + (1 - v_0)(\alpha^*\mu + \beta\mu - |\mu|^2)] \\ &\quad \times \exp(-\tfrac{1}{2}|\alpha|^2 - \tfrac{1}{2}|\beta|^2)\end{aligned} \tag{22}$$

where $v_0 = e^{-w}$. Setting $\mu = 0$ in (22), we obtain

$$\langle\alpha | \rho_0 | \beta\rangle = (1 - v_0) \exp(v_0\alpha^*\beta - \tfrac{1}{2}|\alpha|^2 - \tfrac{1}{2}|\beta|^2) \tag{23}$$

There also exists a coherent "diagonal" representation of ρ_1 and ρ_0, which may be useful in some applications.[18,19] In this representation,

$$\rho_1 = \int [\exp(-|\alpha - \mu|^2/N)]\, |\alpha\rangle\langle\alpha|\, (d^2\alpha/\pi N)$$

ρ_0 is obtained again by setting $\mu = 0$ in ρ_1 above.

If we expand the detection operator in terms of number states $|\, n\rangle$, we have

$$\pi = \sum_{n,m} |\, n\rangle\langle n \,|\, \pi \,|\, m\rangle\langle m \,| = \sum_{n,m} \pi_{nm} \,|\, n\rangle\langle m \,|$$

Then the matrix element $\langle\beta \,|\, \pi \,|\, \alpha\rangle$ is given by using (11),

$$\langle\beta \,|\, \pi \,|\, \alpha\rangle = \left\{\sum_{n,m} [\pi_{nm}/(n!m!)^{1/2}]\, \beta^{*n}\alpha^{m}\right\} \exp(-\tfrac{1}{2} \,|\, \alpha \,|^2 - \tfrac{1}{2} \,|\, \beta \,|^2)$$

Since $|\, \pi_{nm} \,| \leqslant 1$, the sum in the braces is absolutely convergent for all finite β^* and α and thus defines an entire function of two variables, $\pi(\beta^*, \alpha)$. To find the detection operator, we must therefore search for the function $\pi(\beta^*, \alpha)$ that maximizes the integral

$$\begin{aligned} \mathrm{Tr}(\rho_1 - \lambda\rho_0)\, \pi &= \iint \langle\alpha \,|\, \rho_1 - \lambda\rho_0 \,|\, \beta\rangle\langle\beta \,|\, \pi \,|\, \alpha\rangle(d^2\alpha/\pi)(d^2\beta/\pi) \\ &= \iint [R_1(\alpha^*, \beta) - \lambda R_0(\alpha^*, \beta)]\, \pi(\beta^*, \alpha) \\ &\quad \times [\exp(- \,|\, \alpha \,|^2 - \,|\, \beta \,|^2)](d^2\alpha/\pi)(d^2\beta/\pi) \end{aligned} \tag{24}$$

$R_1(\alpha^*, \beta)$ and $R_0(\alpha^*, \beta)$ are known entire functions of two variables α^* and β, where

$$R_1(\alpha^*, \beta) = (1 - v_0) \exp[v_0\alpha^*\beta + (1 - v_0)(\alpha^*\mu + \beta\mu^* - |\, \mu \,|^2)] \tag{25}$$

and

$$R_0(\alpha^*, \beta) = (1 - v_0) \exp[v_0\alpha^*\beta] \tag{26}$$

The constraint condition $0 \leqslant \langle f \,|\, \pi \,|\, f\rangle \leqslant 1$ for all $|\, f\rangle$ of norm 1 becomes

$$0 \leqslant \iint [f(\beta^*)]^*\, \pi(\beta^*, \alpha) f(\alpha^*)[\exp(- \,|\, \alpha \,|^2 - \,|\, \beta \,|^2)](d^2\beta/\pi)(d^2\alpha/\pi) \leqslant 1$$

for all entire functions $f(\alpha^*)$ such that

$$\int |\, f(\alpha^*)|^2(\exp - \,|\, \alpha \,|^2)(d^2\alpha/\pi) = 1$$

The arbitrariness of $f(\alpha^*)$ in the constraint condition prevents us from applying the usual techniques of optimizing the functionals, and the problem of finding $\pi(\beta^*, \alpha)$ appears very difficult.

We note in passing that the problem is easier if the phase of the coherent signal is a random variable. We may assume the uniform distribution in $[0, 2\pi]$ for the phase, and the density operator for the case is ρ_1 of (20) averaged over $[0, 2\pi]$. Then, the new density operators ρ_1 and ρ_0 commute and since the number operator N commutes with both, it suffices to measure N. Helstrom(2) solved this problem and published curves of the result.(22) Liu(21) extended his results and derived the bounds on the error probability.

To solve the problem when the phase is known, it appears we must resort to some approximation. The next section discusses one approximation particularly applicable to optical signals.

4. SOLUTION FOR OPTICAL FREQUENCIES

We recall from Section 2 that our detection problem is solved completely if we can solve the eigenvalue equation

$$(\rho_1 - \lambda\rho_0) \mid \eta\rangle = \eta \mid \eta\rangle$$

where ρ_1 and ρ_0 are given positive trace-class operators and λ is any positive number. Since both ρ_1 and ρ_0 are infinite-dimensional, we may try to approximate them by their finite-dimensional counterparts ρ_{1M} and ρ_{0M}. A convenient coordinate for computing the matrix elements of ρ_1 and ρ_0 is the set of number states $\mid n\rangle$. The operator ρ_0 is diagonal in the number states:

$$\langle n \mid \rho_0 \mid m\rangle = (1 - v_0)\, v_0{}^n\, \delta_{nm} \tag{27}$$

To compute the matrix element $\langle n \mid \rho_1 \mid m\rangle$, we use the expansion of $\mid n\rangle$ in terms of coherent states:

$$\begin{aligned}\mid n\rangle &= \int \mid \alpha\rangle\langle\alpha \mid n\rangle(d^2\alpha/\pi)\\ &= \int \mid \alpha\rangle \sum_{n=0}^{\infty} [\alpha^{*n}/(n!)^{1/2}](\exp -\tfrac{1}{2} \mid \alpha \mid^2)(d\alpha/\pi)\end{aligned}$$

Then,

$$\begin{aligned}\langle n \mid \rho_1 \mid m\rangle = \iint \langle\alpha \mid \rho_1 \mid m\rangle\\ \times \sum_{n,m} [\alpha^n\alpha^{*m}/(n!m!)^{1/2}][\exp(-\tfrac{1}{2} \mid \alpha \mid^2 - \tfrac{1}{2} \mid \beta \mid^2)](d^2\alpha/\pi)(d^2\beta/\pi)\end{aligned}$$

Substituting for $\langle\alpha \mid \rho_1 \mid \beta\rangle$ from (22), expanding $\exp(v_0\alpha^*\beta)$ in series, and using the reproducing formula (16), we obtain

$$\begin{aligned}\langle n \mid \rho_1 \mid m\rangle = (1 - v_0)(\exp -\mid \mu \mid^2)(n!/m!)^{1/2}\, [v_0{}^n/(n-m)!](\mu/N)^{n-m}\\ \times M(n+1, n-m+1; v_0 \mid \mu \mid^2/N^2)\end{aligned}$$

where $M(a, b; z)$ is the confluent hypergeometric function. Using the Kummer transformation $M(a, b; z) = e^z M(b-a, b; -z)$, we can also write the above as[2]

$$\begin{aligned}\langle n \mid \rho_1 \mid m\rangle = (1 - v_0)\, v_0{}^n\{\exp[-\mid \mu \mid^2/(N+1)]\}(m!/n!)^{1/2}\\ \times (\mu/N)^{n-m}\, L_m^{(n-m)}(-\mid \mu \mid^2/N(N+1))\end{aligned} \tag{28}$$

where $v_0 = N/(N+1)$, and $L_l^{(p)}(x)$ is the generalized Laguerre polynominal.

Equations (27) and (28) give the matrix representation of ρ_0 and ρ_1 on the number states. By truncating the matrices after M dimensions, we obtain ρ_{1M} and ρ_{0M}. The problem is ready for machine calculation of eigenvalues and eigenvectors of

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[2] C. W. Helstrom, private communication.

$\rho_{1M} - \lambda\rho_{0M}$. There are, however, some difficulties with this approach. The first is that we cannot readily find a bound on the error we make by truncating the matrices. It is clear that by increasing M we obtain more accuracy, but it is not apparent what M is sufficiently large. Second, for conditions of communication at optical frequencies, v_0, hence N, is very small. Looking at (27) and (28), we foresee some difficulty in computing the matrix element accurately for small N. Furthermore, there are always inaccuracies in machine computation of eigenvalues and eigenvectors, especially for large matrices, and this is in conflict with the requirement for large matrices.

For these reasons, we abandoned the direct finite-dimensional approach and decided to look for a faster and more accurate algorithm especially suited for optical frequencies, where v_0 is small. An algorithm with such characteristics is derived next.

The eigenvalue equation $(\rho_1 - \lambda\rho_0) \mid \eta\rangle = \eta \mid \eta\rangle$ can be converted into an integral equation by premultiplying by $\langle\alpha\mid$ and expanding $\mid\eta\rangle$ in terms of $\mid\beta\rangle$. That is, if we put

$$\begin{aligned} \mid \eta\rangle &= \int \mid \beta\rangle\langle\beta \mid \eta\rangle(d^2\beta/\pi) \\ &= \int \mid \beta\rangle\, F(\beta^*)[\exp(-\tfrac{1}{2} \mid \beta \mid^2)](d^2\beta/\pi) \end{aligned}$$

where $F(\beta^*) = \langle\beta \mid \eta\rangle \exp(\frac{1}{2} \mid \beta \mid^2)$, then we have

$$\int [\langle\alpha \mid \rho_1 \mid \beta\rangle - \lambda\langle\alpha \mid \rho_0 \mid \beta\rangle]\langle\beta \mid \eta\rangle(d^2\beta/\pi) = \eta\langle\alpha \mid \eta\rangle$$

This is an integral equation of the form

$$\int K(\alpha^*, \beta)\, F(\beta^*)(\exp - \mid \beta \mid^2)(d^2\beta/\pi) = \eta F(\alpha^*) \tag{29}$$

where $K(\alpha^*, \beta) = R_1(\alpha^*, \beta) - \lambda R_0(\alpha^*, \beta)$ and where $R_1(\alpha^*, \beta)$ and $R_0(\alpha^*, \beta)$ are given by (25) and (26), respectively.

We now wish to solve (29) for eigenvalue and eigenfunction $F(\alpha^*)$, which must belong to the Hilbert space of analytic functions $\mathscr{F}$. The key to the derivation is note that both $R_1(\alpha^*, \beta)$ and $R_0(\alpha^*, \beta)$ contain reproducing kernels of the form $\exp \xi^*\beta$. Thus, we can immediately integrate (29) by the reproducing formula (16). The result is a functional equation for $F(\alpha^*)$:

$$\{\exp[(1 - v_0)(\alpha^* - \mu^*)\, \mu]\}\, F(v_0\alpha^* + (1 - v_0)\, \mu^*) - \lambda F(v_0\alpha^*) = \epsilon F(\alpha^*) \tag{30}$$

where $\epsilon = \eta/(1 - v_0)$.

We still cannot solve this equation in general but, because F is analytic, we can make progress. Our interest in the solution is primarily for the optical frequencies, where v_0 is small. Therefore, we shall expand (30) in powers of v_0 and attempt to solve simpler functional equations of low order in v_0. The expansion gives

$$\begin{aligned} &[\exp(\alpha^*\mu - \mid \mu \mid^2)] \sum_{n=0}^{\infty} (v_0{}^n/n!)\, G^{(n)}(\mu^*)(\alpha^* - \mu^*)^n \\ &\qquad - \lambda \sum_{n=0}^{\infty} (v_0{}^n/n!)\, F^{(n)}(0)\, \alpha^{*n} = \epsilon F(\alpha^*) \end{aligned} \tag{31}$$

where $G^{(n)}(\mu^*)$ is a constant given by

$$G^{(n)}(\mu^*) = \sum_{k=0}^{n} \binom{n}{k} (-\mu)^k F^{(n-k)}(\mu^*) \tag{32}$$

Finding the constants $G^{(n)}(\mu^*)$ and $F^{(n)}(0)$ is equivalent to solving the functional equation (30). Although we are unable to solve for all constants, we can truncate the series after n_0 terms and solve for the constants in the truncated series. Since the series converges, the error is of the order $v_0^{(n_0+1)}$ and since $v_0 \ll 1$ for optical frequencies, the solution for the truncated series can be an excellent approximation. The clue to finding the constants $G^{(n)}(\mu^*)$ and $F^{(n)}(0)$ comes from the zeroth-order ($n = 0$) solution. Note that the zeroth-order solution is equivalent to setting $v_0 = 0$, or since $v_0 = \exp(-hf/kT)$, it is the limiting solution for very high frequency or very low temperature. Setting $v_0 = 0$, we obtain

$$[\exp(\alpha^*\mu - |\mu|^2)] F(\mu^*) - \lambda F(0) - \epsilon F(\alpha^*) \tag{33}$$

By alternately setting $\alpha^* = \mu^*$ and $\alpha^* = 0$, we arrive at a two-by-two matrix eigenvalue equation

$$\begin{bmatrix} 1 & -\lambda \\ \exp(-|\mu|^2) & -\lambda \end{bmatrix} \begin{bmatrix} F(\mu^*) \\ F(0) \end{bmatrix} = \epsilon \begin{bmatrix} F(\mu^*) \\ F(0) \end{bmatrix} \tag{34}$$

The equation for eigenvalue ϵ is

$$\epsilon^2 + (\lambda - 1)\epsilon - \lambda q = 0 \tag{35}$$

where $q = 1 - \exp -|\mu|^2$. It is clear that for the case $v_0 = 0$ there are only two eigenvalues, one positive and the other negative:

$$\epsilon_\pm = [(1 - \lambda)/2] \pm R \tag{36}$$

where

$$R = \{[(1 - \lambda)/2]^2 + \lambda q\}^{1/2} \tag{37}$$

Substituting the result from (34) into (33), we see that the eigenfunction for $v_0 = 0$ is

$$F^{(0)}(\alpha^*) = (C_\pm/\epsilon_\pm)[(\epsilon_\pm + \lambda)(\exp \alpha^*\mu) - \lambda]$$

where $C_\pm$ is the constant determined from the normalization condition $1 = (F^{(0)}, F^{(0)})$ and is given by $C_\pm = [(\epsilon_\pm - q)/\pm 2R]^{1/2}$.

We digress a little to obtain an expression for ρ_1 and ρ_0 when $v_0 = 0$. Recall $v_0 = e^{-w}$ and $\rho_1 = (1 - e^{-w}) \exp(-wb^+b)$, where $b = a - \mu$. Then, as $v_0 \to 0$ (Louisell,[16] p. 248),

$$\rho_1 |_{v_0=0} = \lim_{w\to\infty} (1 - e^{-w}) \exp(-wb^+b) = |0\rangle_b\, {}_b\langle 0|$$

where $|0\rangle_b$ is the eigenstate of the number operator $N_b = b^+b$ corresponding to number zero. But since $b|0\rangle_b = 0$, $a|0\rangle_b = \mu|0\rangle_b$, and from (10) and the uniqueness of coherent states, we obtain $| 0\rangle_b = | \mu\rangle$. Therefore, $\rho_1(v_0 = 0) = | \mu\rangle\langle\mu |$ and $\rho_0(v_0 = 0) = | 0\rangle\langle 0 |$.

The probability of detection Q_d is given by $Q_d = \langle\epsilon_+ | \rho_1 | \epsilon_+\rangle$, where $| \epsilon_+\rangle$ is the eigenvector corresponding to the positive eigenvalue ϵ_+. After some algebra, we obtain

$$Q_d = |\langle\epsilon_+ | \mu\rangle|^2 = (\epsilon_+ + \lambda q)/2R \tag{38}$$

Similarly, the probability of false alarm is given by

$$Q_0 = |\langle\epsilon_+ | 0\rangle|^2 = (\epsilon_+ - q)/2R \tag{39}$$

Expressions (38) and (39) are identical to those due to Helstrom,[4] who obtained them by solving the eigenvalue equation

$$H_0 | \eta\rangle = (| \mu\rangle\langle\mu | - \lambda | 0\rangle\langle 0 |) | \eta\rangle = \eta | \eta\rangle$$

We have thus far enumerated only two solutions of the above equation. They are given by

$$| \eta_\pm\rangle = (C_\pm/\eta_\pm)\{(\epsilon_\pm + \lambda)[\exp(| \mu |^2/2)] | \mu\rangle - \lambda | 0\rangle\}$$

Any other eigenvector $| \eta_k\rangle$ must be orthogonal to both $| \eta_+\rangle$ and $| \eta_-\rangle$. A simple calculation shows that $| \eta_k\rangle$ must be orthogonal to both $| \mu\rangle$ and $| 0\rangle$. Therefore, the contribution from $| \eta_k\rangle$ to the probability of detection and to the probability of false alarm is zero. Moreover, since the range space of H_0 is spanned by $| \mu\rangle$ and $| 0\rangle$, $| \eta_k\rangle$ must belong to the null space of H_0, i.e., its eigenvalues must be zero. This completes the solution for the case of $v_0 = 0$.

Having solved the problem exactly for $v_0 = 0$, we may try to apply the known techniques of perturbation theory to obtain solutions for small v_0. For example, the first-order expansion of $\rho_1 - \lambda\rho_0$ is

$$\begin{aligned} H_1 &= H_0 + v_0 V \\ &= | \mu\rangle\langle\mu | - \lambda | 0\rangle\langle 0 | + v_0\{(a^+ - \mu^*) | \mu\rangle\langle\mu | (a - \mu) - \lambda a^+ | 0\rangle\langle 0 | a\} \end{aligned}$$

where V is the expression in the braces. The first-order correction to eigenvalues $\epsilon_\pm$ is given by $\langle\eta_\pm | V | \eta_\pm\rangle$. But, in addition to $\epsilon_\pm$, there are infinitely many eigenvalues at zero that must be perturbed. The eigenvectors corresponding to zero eigenvectors must be chosen out of the subspace orthogonal to $| \eta_\pm\rangle$. But there are infinitely many such vectors and each choice results in a different correction and this is clearly untenable. In short, the usual techniques of perturbation theory are not useful, because of the infinite-fold degeneracy at zero eigenvalue. Removing the degeneracy by diagonalizing is as difficult as solving the original problem. Fortunately, an approach through functional equation (31) is available and the zeroth-order solution gives us a hint as to how we might proceed.

We recall that we solved the zeroth-order functional equation (33) by alternately setting $\alpha^* = \mu^*$ and then $\alpha^* = 0$. To solve the first-order equation, we must determine four constants: $G^{(0)}$, $G^{(1)}$, $F(0)$ and $F^{(1)}(0)$. The first-order equation is

$$\begin{aligned} &[\exp(\alpha^*\mu - | \mu |^2)][G^{(0)}(\mu^*) + v_0 G^{(1)}(\mu^*)(\alpha^* - \mu^*)] \\ &\quad - \lambda[F(0) + v_0 F^{(1)}(0)\, \alpha^*] = \epsilon F(\alpha^*) \end{aligned} \tag{40}$$

By substituting $\alpha^* = \mu^*$ and $\alpha^* = 0$, we see that two equations relating the four constants are obtained. Two other equations are readily derived by first differentiating (40) with respect to α^* and then setting $\alpha^* = \mu^*$ and $\alpha^* = 0$ alternately.

It is clear how we may generalize this procedure to obtain the n_0th-order solution. We differentiate (31) k times, $0 \leqslant k \leqslant n_0$. The result is

$$\sum_{n=0}^{\infty} v_0{}^n G^{(n)}(\mu^*) \left[\sum_{l=0}^{M} \binom{k}{l} \frac{\mu^{k-l}(\alpha^* - \mu^*)^{n-l}}{(n-l)!}\right] [\exp \mu(\alpha^* - \mu^*)]$$
$$-\lambda \sum_{n=k}^{\infty} \frac{v_0{}^n \alpha^*}{(n-k)!} F^{(n)}(0) = \epsilon \frac{\partial^k}{\partial \alpha^{*k}} F(\alpha^*) \tag{41}$$

where $M = \min(m, k)$. We now set $\alpha^* = \mu^*$ and $\alpha^* = 0$ alternately to obtain two sets of equations; when $\alpha^* = \mu$,

$$\sum_{n=0}^{k} \binom{k}{n} \mu^{k-n} v_0{}^n G^{(n)}(\mu^*) - \lambda \sum_{n=k}^{\infty} \frac{v_0{}^n \mu^{*n-k}}{(n-k)!} F^{(n)}(0) = \epsilon F^{(k)}(\mu^*) \tag{42}$$

and when $\alpha^* = 0$,

$$(\exp - |\mu|^2) \sum_{n=0}^{\infty} v_0{}^n G^{(n)}(\mu^*) \left[\sum_{l=0}^{M} \binom{k}{l} \frac{\mu^{k-l}(-\mu^*)^{n-l}}{(n-l)!}\right] - \lambda v_0{}^k F^{(k)}(0) = \epsilon F^{(k)}(0) \tag{43}$$

We see by writing out (42) and (43), with help from (32), that the unknowns $F(\mu^*)$, $F(0)$, $F^{(1)}(\mu^*)$, $F^{(1)}(0)$,..., $F^{(n)}(\mu^*)$, $F^{(n)}(0)$,... are the components of an eigenvector corresponding to eigenvalue ϵ.

Thus, the problem again reduces to solving an infinite-dimensional matrix eigenvalue equation, not unlike the earlier matrix equation on the number states. We have gained something through our work, however, when we truncate the matrix given by (42) and (43) after n_0 dimensions, all terms neglected are of order $v_0^{(n_0+1)}$ and higher. We cannot claim the same for the number representation. Moreover, since v_0 is small for most cases of our interest, this algorithm gives us a good bound on the error. Another advantage of the present algorithm over the number representation is that for small v_0 the matrix can be kept small. It is clear that in order to keep the error below $v_0^{(n_0+1)}$ the matrix must be $2(n_0 + 1)$ by $2(n_0 + 1)$, $0 \leqslant n_0 < \infty$. For optical frequency applications, n_0 of two or three should be adequate, and for these small matrices the eigenvalues can generally be calculated quite accurately.

The components of the eigenvector must be normalized according to $(F, F) = 1$. Using (31), we have

$$\epsilon = \sum_{n=0}^{\infty} (v_0{}^n/n!)[G^{(n)}(\mu^*)]^* \int (\alpha - \mu)^n [\exp \mu^*(\alpha - \mu)]\, F(\alpha^*)(\exp - |\alpha|^2)(d^2\alpha/\pi)$$
$$-\lambda \sum_{n=0}^{\infty} (v_0{}^n/n!)[F^{(n)}(0)]^* \int \alpha^n F(\alpha^*)(\exp - |\alpha|^2)(d^2\alpha/\pi) \tag{44}$$

The first integral, on expanding $F(\alpha^*)$ as in (2.8), becomes

$$I_n = \int (\alpha - \mu)^n [\exp \mu^*(\alpha - \mu)] \sum_{k=0}^{\infty} [f_k/(k!)^{1/2}] \, \alpha^{*k} (\exp - |\alpha|^2)(d^2\alpha/\pi)$$

$$= \sum_{k=0}^{\infty} [f_k/(k!)^{1/2}](\partial^n/\partial\mu^{*n}) \int [\exp \mu^*(\alpha - \mu)] \, \alpha^{*k} (\exp - |\alpha|^2)(d^2\alpha/\pi)$$

The last integral can be evaluated by the reproducing formula (16). The result is

$$I_n = \sum_{k=0}^{\infty} [f_k/(k!)^{1/2}](\partial^n/\partial\mu^{*n}) \, e^{-\mu^*\mu} \mu^{*k}$$

$$= \sum_{k=0}^{\infty} [f_k/(k!)^{1/2}](\partial^n/\partial\alpha^{*n}) \, e^{\alpha^*\mu} \alpha^{*k} \,|_{\alpha^*=\mu^*}$$

$$= (\partial^n/\partial\alpha^{*n}) \, e^{-\alpha^*\mu} \sum_{k=0}^{\infty} [f_n/(k!)^{1/2}] \, \alpha^{*k} \,|_{\alpha^*=\mu^*}$$

$$= (\partial^n/\partial\alpha^{*n}) \, e^{\alpha^*\mu} F(\alpha^*)|_{\alpha^*=\mu^*}$$

It is easy to show by induction that $I_n = (\exp - |\mu|^2) \, G^{(n)}(\mu^*)$, where $G^{(n)}(\mu^*)$ is as given by (32). The second integral of (44) may be similarly computed. The result is

$$J_n = \int \alpha^n F(\alpha^*)(\exp - |\alpha|^2)(d^2\alpha/\pi) = F^{(n)}(0)$$

Therefore, the normalization condition (44) becomes, for the kth eigenvalue,

$$\epsilon_k = \sum_{n=0}^{\infty} (v_0{}^n/n!)\{|G^{(n)}(\mu^*)|^2(\exp - |\mu|^2) - \lambda \, |F^{(n)}(0)|^2\} \tag{45}$$

For illustrative purposes, we write out (42), (43), and (45) in matrix form for $n_0 = 1$. For this first-order calculation, we have from (32)

$$X_G = \begin{bmatrix} G^{(0)}(\mu^*) \\ F^{(0)}(0) \\ G^{(1)}(\mu^*) \\ F^{(1)}(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\mu & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F^{(0)}(\mu^*) \\ F^{(0)}(0) \\ F^{(1)}(\mu^*) \\ F^{(1)}(0) \end{bmatrix}$$

The eigenvalue equation is

$$\begin{bmatrix} 1 & -\lambda & 0 & -\lambda v_0 \mu \\ \exp(-\mu^2) & -\lambda & -v_0\mu \exp(-\mu^2) & 0 \\ \mu & 0 & v_0 & -\lambda v_0 \\ \mu \exp(-\mu^2) & 0 & v_0 \exp(-\mu^2) & -\lambda v_0 \\ & & X(1 - \mu^2) & \end{bmatrix} X_G = \epsilon \begin{bmatrix} F^{(0)}(\mu) \\ F^{(0)}(0) \\ F^{(1)}(\mu) \\ F^{(1)}(0) \end{bmatrix} \tag{46}$$

where we have set the phase of μ at zero without loss in generality. The normalization condition (46) becomes

$$X_G^T \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & v_0 & 0 \\ 0 & 0 & 0 & -v_0\lambda \end{bmatrix} X_G = \epsilon$$

Incidentally, we note that eigenvalue ϵ must be real, despite the fact that the matrix in (45) is not Hermitean, since the kernel in the integral equation (29) is symmetric after any truncation. For the first-order calculation, we can show more. The eigenvalue equation is a fourth-order polynomial

$$\begin{aligned} &\epsilon^4 + (\lambda - 1)(1 + v_0)\,\epsilon^3 + \{(\lambda - v_0)(v_0\lambda - 1) - 2v_0\lambda q + \lambda(1 - q)[1 - v_0 + v_0 N_s]\}\,\epsilon^2 \\ &\quad + v_0\lambda(\lambda - 1)[N_s(1 - q)(1 - v_0 + v_0 N_s) - q(1 + v_0)]\,\epsilon \\ &\quad + v_0^2\lambda^2\{q^2 - (1 - q)\,N_s^2\} = 0 \end{aligned} \tag{47}$$

where $N_s = \mu^2$. The roots of (47) are given in Fig. 1 as function of the perturbing parameter v_0. We note that when v_0 is very small the two eigenvalues near zero vanish while the outer eigenvalues approach those specified by (36). We can show this more clearly by letting $v_0 = 0$ in (47). There results

$$\epsilon^2[\epsilon^2 + (\lambda - 1)\,\epsilon - \lambda q] = 0$$

The four roots are 0, 0, and $\epsilon_\pm$ as given by (36).

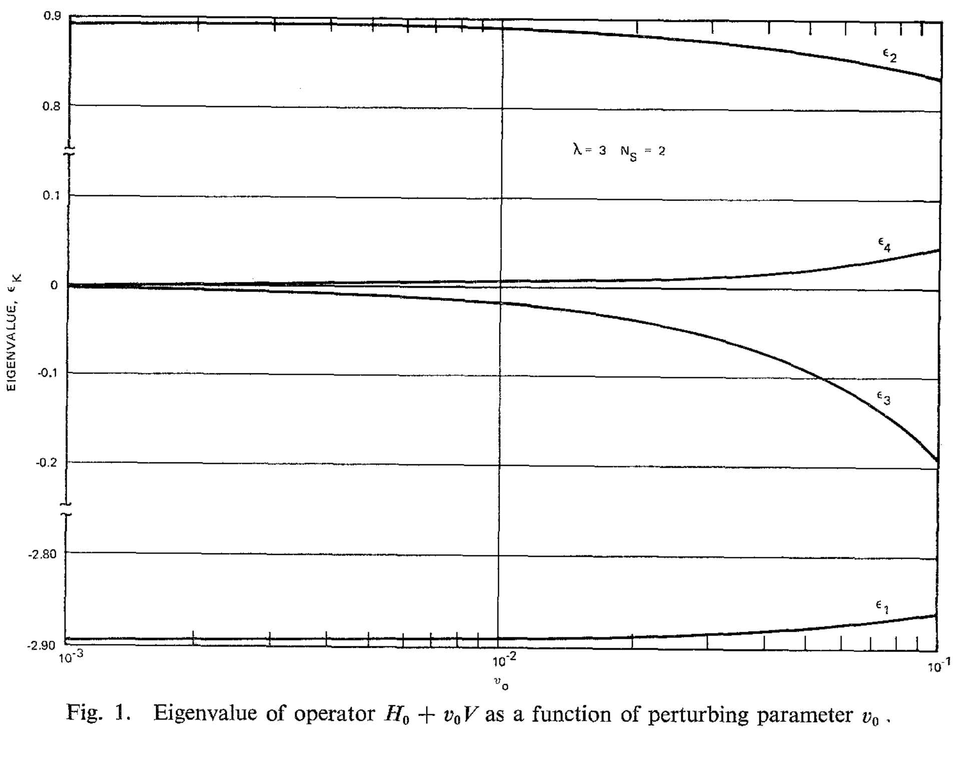

Fig. 1. Eigenvalue of operator $H_0 + v_0 V$ as a function of perturbing parameter v_0.

It may be of interest to compute the eigenvector $| \eta\rangle$ of operator $\rho_1 - \lambda\rho_0$. From (14) and (31), we have

$$| \eta\rangle = (1/\epsilon) \sum_{n=0}^{\infty} (v_0{}^n/n!) \Big(G^{(n)}(\mu^*) \times \int | \alpha\rangle(\alpha^* - \mu^*)^n\{\exp[\mu(\alpha^* - \mu^*) - (| \alpha |^2/2)]\}(d^2\alpha/\pi) - \lambda F^{(n)}(0) \int | \alpha\rangle \alpha^{*^n} [\exp(- | \alpha |^2/2)](d^2\alpha/\pi)\Big)$$

Let us consider the second integral first. We note that

$$\langle\alpha | a^{+^n} | 0\rangle = \alpha^{*^n}\langle\alpha | 0\rangle = \alpha^{*^n} \exp(-| \alpha |^2/2)$$

so that the integral is, by an elementary property of the creation operator,

$$\int | \alpha\rangle(d^2\alpha/\pi)\langle\alpha | a^{+^n} | 0\rangle = a^{+^n} | 0\rangle = (n!)^{1/2} | n\rangle$$

The first integral can be rewritten as

$$[\exp(- | \mu |^2/2)] \int | \alpha\rangle(\alpha^* - \mu^*)^n\langle\alpha | \mu\rangle(d^2\alpha/\pi)$$

using the relation $\langle\alpha | \mu\rangle = \exp[\alpha^*\mu - (| \alpha |^2/2) - (| \mu |^2/2)]$. But $\langle\alpha | (a^+ - \mu^*)^n = (\alpha^* - \mu^*)^n \langle\alpha |$, so that the integral becomes $[\exp(-| \mu |^2/2)](a^+ - \mu^*)^n | n\rangle$. The kth eigenvector of $\rho_1 - \lambda\rho_0$ is therefore

$$| \eta_k\rangle = (1/\epsilon_k) \sum_{n=0}^{\infty} (v_0{}^n/n!)[G_k^{(n)}(\mu^*)[\exp(- | \mu |^2/2)](a^+ - \mu^*)^n | \mu\rangle - \lambda F_k^{(n)}(0)(n!)^{1/2} | n\rangle] \tag{48}$$

The optimum detection operator π_0 can be constructed from (48) but it is not obvious how the expression for π_0 can be simplified. A procedure equivalent to measuring π_0 is to measure the operator $\rho_1 - \lambda\rho_0$ and decide that a signal is present if a positive eigenvalue is obtained.

It remains to calculate the probability of detection Q_d and the probability of false alarm Q_0. We recall that

$$\mathrm{Tr}\, \rho_1\pi - \lambda\, \mathrm{Tr}\, \rho_0\pi = Q_d - \lambda Q_0 = \sum_{k:\eta_k \geqslant 0} \eta_k$$

Therefore,

$$Q_d = (1 - v_0) \sum_{k:\epsilon_k \geqslant 0} \epsilon_k + \lambda Q_0 \tag{49}$$

Since we have now computed eigenvalues ϵ_k, it suffices to calculate Q_0. We recall

$$Q_0 = \mathrm{Tr}\, \rho_0\pi = \sum_{k:\epsilon_k \geqslant 0} \langle\epsilon_k | \rho_0 | \epsilon_k\rangle$$

Making use of (23), (48), and the reproducing formula, we obtain

$$Q_0 = (1 - v_0) \sum_{k:\epsilon_k \geqslant 0} \left\{ \sum_{n=0}^{\infty} (v_0{}^n/n!) \left| \int \beta^n F_k(\beta^*)(\exp - |\beta|^2)(d^2\beta/\pi) \right|^2 \right\}$$

$$= (1 - v_0) \sum_{k:n_k \geqslant 0} \left\{ \sum_{n=0}^{\infty} (v_0{}^n/n!) \, | F_k^{(n)}(0)|^2 \right\} \tag{50}$$

Some numerical results are given in Fig. 2–6. Values of Q_d and Q_0 for these figures were computed by a third-order perturbation, retaining terms of order up to $v_0{}^3$. Here, $n_0 = 3$; therefore, the size of the corresponding matrix eigenvalue equation is eight by eight.

In Fig. 2, the probability of error P_e is given by

$$P_e = \tfrac{1}{2}(1 - Q_d) + \tfrac{1}{2}Q_0$$

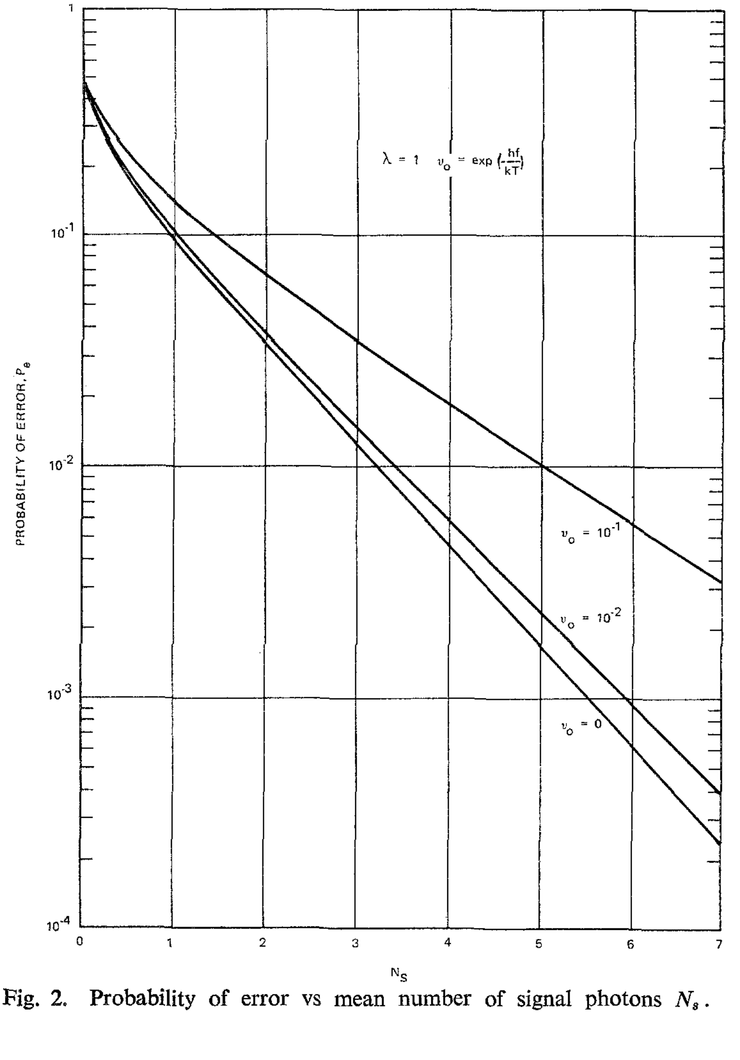

Fig. 2. Probability of error vs mean number of signal photons N_s.

We note that, for $v_0 < 10^{-2}$, P_e is very close to the corresponding values of P_e for $v_0 = 0$. We also observe in Figs. 3–5 that, if v_0 is smaller than 10^{-2}, its Q_d is only slightly less than that for $v_0 = 0$. It appears that, for v_0 less than 10^{-2}, Q_d and Q_0 for $v_0 = 0$, as given by (38) and (39), are good approximations to the true values.

Figure 6 gives an interesting performance curve. The most promising application of the laser in communication systems appears to be in the space environment. Accordingly, we consider a binary laser communication system between a satellite station around the earth and a station on the moon. The messages are coded into 0's and 1's, where a 1 is transmitted by sending a laser pulse and a 0 by sending nothing. We suppose 0's and 1's to be equally probable and an ideal receiver to measure the signal from a carbon dioxide laser. We further assume that the earth satellite station sees lunar day as background and the moon station sees the sunlit earth as background. Under these conditions, the signal wavelength is 10.6 μm and background temperatures are 373°K for the moon and 300°K for the earth.[23] Figure 6 gives the lower bound on the error probability in this communication environment. It is a bound in the sense that to achieve a lower probability than indicated is tantamount to violating the laws of physics.

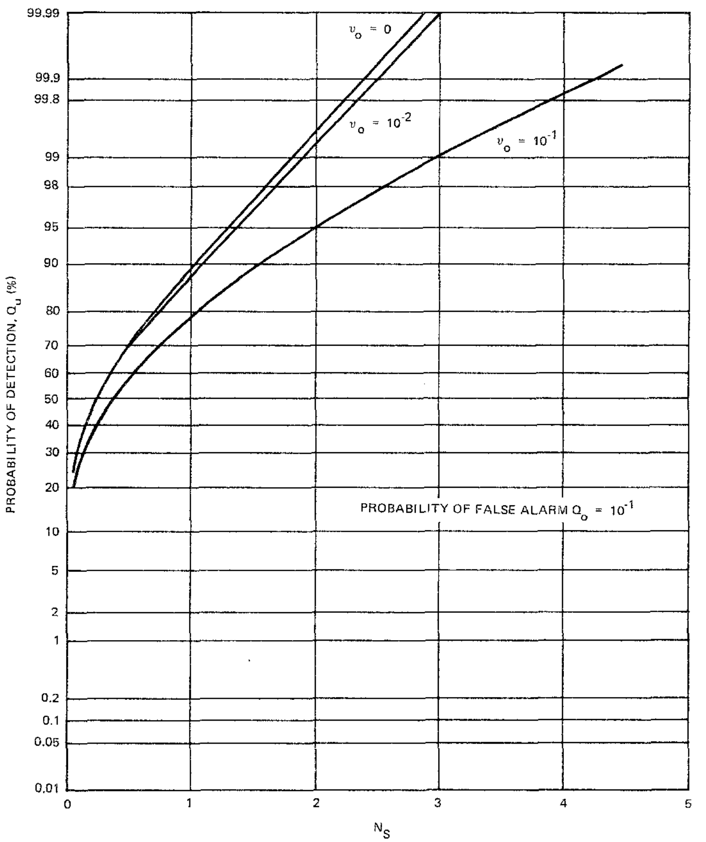

Fig. 3. Probability of detection vs mean number of signal photons; $Q_0 = 10^{-1}$.

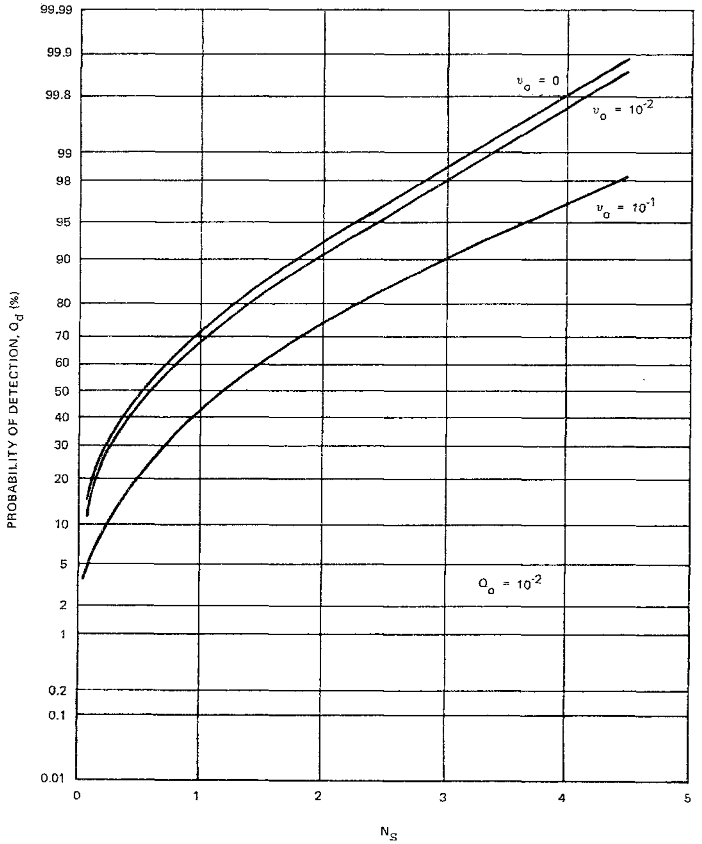

Fig. 4. Probability of detection vs mean number of signal photons; $Q_0 = 10^{-1}$.

For comparison, we included the error probability curve of a classical "on–off" binary channel. Its error probability is given by (Helstrom,[8] p. 190)

$$P_e = \text{erfc}(D/2)$$

where erfc(.) is the complementary error function. In the classical limit of $hf \ll kT$, N given by the Planck formula $N = [\exp(hf/kT) - 1]^{-1}$ becomes kT/nf and the signal-to-noise ratio D^2 becomes $D^2 = 2E/kT$, where $E = N_s hf$ is the energy of the signal.

5. DETECTOR THAT MAXIMIZES SIGNAL-TO-NOISE RATIO

In classical detection theory, when the detector specified by the Neyman–Pearson criterion is too complicated to implement, we sometimes seek a detector that maximizes an appropriately defined signal-to-noise ratio. For example, if U is a sufficient statistic of the data, we may define a signal-to-noise ratio by

$$D^2 = (E_\theta U - E_0 U)^2 / Var_0 U \tag{51}$$

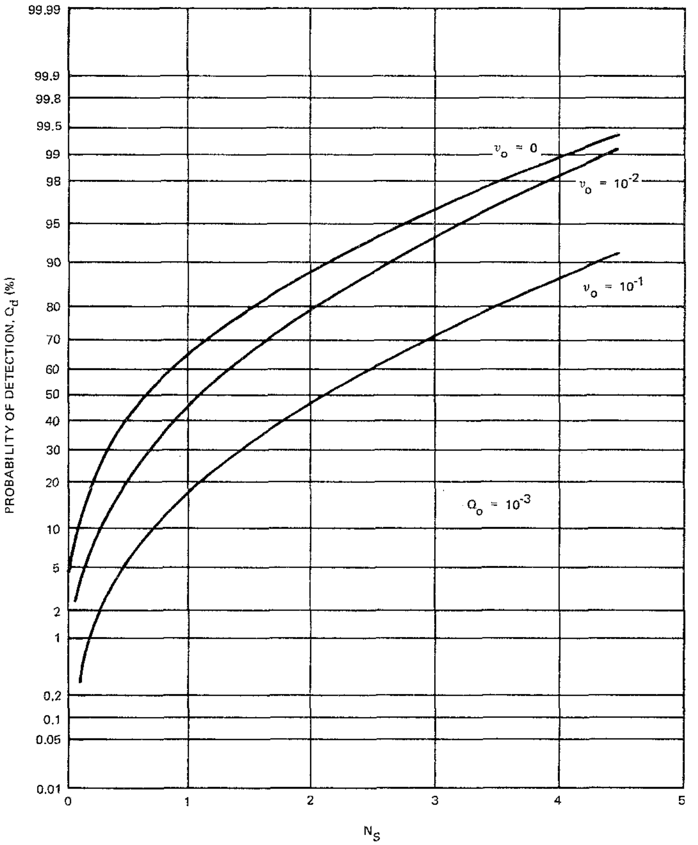

Fig. 5. Probability of detection vs mean number of signal photons; $Q_0 = 10^{-3}$.

where $E_\theta U$ is the conditional expectation of U given that the signal strength is θ; $E_0 U$ is the same with $\theta = 0$; and $\mathrm{Var}_0 U$ is the variance of the statistic in the absence of the signal. A detector that maximizes D^2 in the limit $\theta \to 0$ is called a threshold detector and it is optimum in the sense of maximizing the asymptotic relative efficiency.[24]

Helstrom[2] defined a signal-to-noise ratio for quantum detection problems as

$$D^2 = (\mathrm{Tr}\,\rho_1\pi - \mathrm{Tr}\,\rho_0\pi)^2/[\mathrm{Tr}\,\rho_0\pi^2 - (\mathrm{Tr}\,\rho_0\pi)^2] \tag{52}$$

This is a quantum mechanical analog of (51). The operator π_s that maximizes (52) is given by the solution of the equation

$$2(\rho_1 - \rho_0) = \rho_0\pi_s + \pi_s\rho_0 \tag{53}$$

For proof, we follow Helstrom.[6] Fist, we note that we may put

$$\mathrm{Tr}\,\rho_0\pi_s = 0 \tag{54}$$

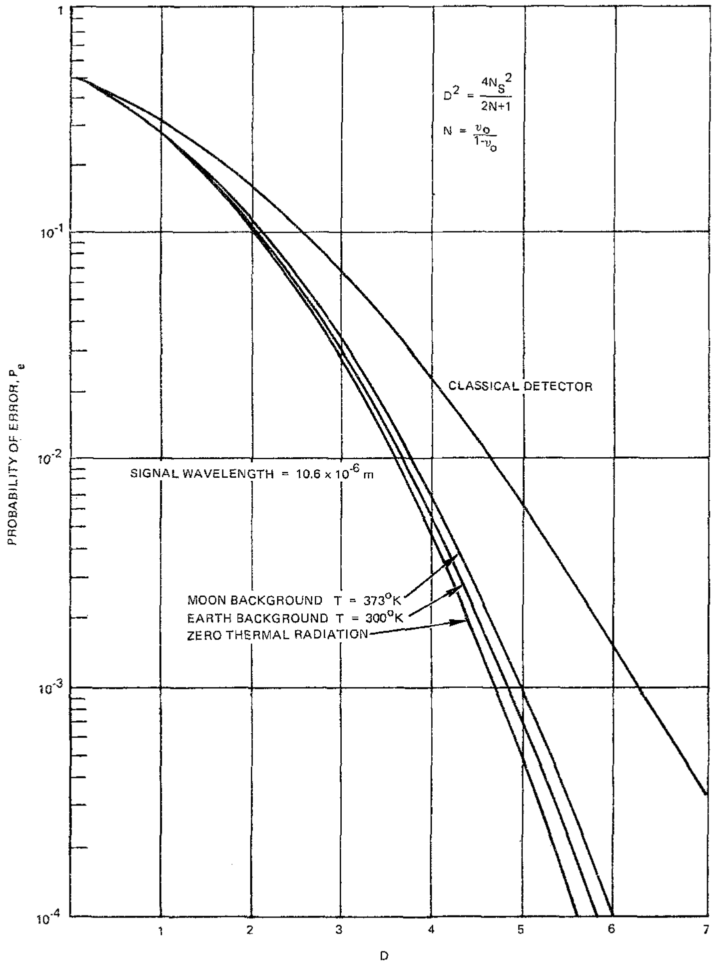

Fig. 6. Probability of error in binary communication vs signal-to-noise ratio D^2.

since a constant operator may be subtracted from π_s without changing D^2. The definition of π_s in (53) shows that $\rho_0 \pi_s$ must be trace-class. π_s satisfies (54) because

$$2(\mathrm{Tr}\ \rho_1 - \mathrm{Tr}\ \rho_0) = 0 = \mathrm{Tr}\ \rho_0 \pi_s + \mathrm{Tr}\ \pi_s \rho_0 = 2\ \mathrm{Tr}\ \rho_0 \pi_s$$

Let π be any other operator such that $\rho_0 \pi^2$ is trace-class. We consider $[\mathrm{Tr}(\rho_1 - \rho_0)\ \pi]^2$. From (53), we find

$$\begin{aligned}
\mathrm{Tr}(\rho_1 - \rho_0)\ \pi^2 &= [\tfrac{1}{2}(\mathrm{Tr}\ \rho_0 \pi_s \pi + \mathrm{Tr}\ \pi_s \rho_0 \pi)]^2 \\
&= [\mathrm{Re}\ \mathrm{Tr}\ \rho_0 \pi_s \pi]^2 \leqslant |\ \mathrm{Tr}\ \rho_0 \pi_s \pi\ |^2 \\
&= \mathrm{Tr}\ \rho_0^{1/2} \pi_s \pi \rho_0^{1/2} \leqslant \mathrm{Tr}(\rho_0^{1/2} \pi_s{}^2 \rho_0^{1/2})\ \mathrm{Tr}(\pi \rho_0 \pi) \\
&= \mathrm{Tr}\ \rho_0 \pi_s{}^2\ \mathrm{Tr}\ \rho_0 \pi^2
\end{aligned}$$

by the Schwarz inequality for traces. Hence, $D^2 \leqslant \operatorname{Tr} \rho_0 \pi^2$, with equality when $\pi_s = \pi$. An alternate expression for the maximum signal-to-noise ratio is obtained by using (53) and (54): $D^2 = \operatorname{Tr} \rho_0 \pi_s^2 = \frac{1}{2} \operatorname{Tr} 2(\rho_1 - \rho_0)\, \pi_s = \operatorname{Tr} \rho_1 \pi_s$.

If θ is the signal strength parameter, the threshold operator π_t is the solution of the equation (Helstrom,[2] p. 275)

$$[\partial \rho_1(\theta)/\partial\theta]|_{\theta=0} = \tfrac{1}{2}(\rho_0 \pi_t + \pi_t \rho_0) \tag{55}$$

Helstrom solved (55) for various cases, among them the coherent known signal case. The π_t thus found maximizes D^2 in the limit of low signal strength.

A more general solution, one that maximizes D^2 for all signal levels, is given by

$$\pi_\theta = 2 \int_0^\infty e^{-\rho_0 y}(\rho_1 - \rho_0)\, e^{-\rho_0 y}\, dy \tag{56}$$

Integrating π_θ by parts, we see that it satisfies (53). We consider several examples below. First, we note that

$$\int_0^\infty e^{-\rho_0 y} 2\rho_0 e^{-\rho_0 y}\, dy = I$$

Therefore, it remains to compute

$$\pi_{s1} = 2 \int_0^\infty e^{-\rho_0 y} \rho_1 e^{-\rho_0 y}\, dy$$

5.1. A Coherent Signal of Unknown Phase

When the phase of the complex parameter μ in (20) is completely unknown, we assume it has the least favorable distribution, which is uniform over the interval $(0, 2\pi)$. We recall that in coherent "diagonal" representation, the density operator under hypothesis H_1 is

$$\rho_1 = \int [\exp(-\mid \alpha - \mu \mid^2/N)] \mid \alpha\rangle\langle\alpha \mid (d^2\alpha/\pi) \tag{57}$$

The density operator for the unknown phase is obtained by averaging (57) with respect to the uniform distribution of $\arg \mu = \psi$ (Helstrom,[2] p. 269),

$$\begin{aligned} \bar{\rho}_1 &= \int_0^{2\pi} \rho_1 (d\psi/2\pi) \\ &= \int_0^{2\pi} (d\psi/2\pi) \int \{\exp[-(\mid \alpha \mid^2 - 2 \mid \alpha \mid \mid \mu \mid \cos(\psi - \theta) + \mid \mu \mid^2)/N]\} \mid \alpha\rangle\langle\alpha \mid (d^2\alpha/\pi N) \\ &= \int \{\exp[-(\mid \alpha \mid + \mid \mu \mid^2)/N]\}\, I_0(2 \mid \alpha \mid \mid \mu \mid/N) \mid \alpha\rangle\langle\alpha \mid (d^2\alpha/\pi N) \end{aligned} \tag{58}$$

where $\theta = \arg \alpha$ and $I_0(.)$ is the modified Bessel function. We can express $\bar{\rho}_1$ in terms of the number states $\mid n\rangle$ as follows: From (22), we have that

$$\mid \alpha\rangle\langle\alpha \mid = [\exp(-\mid \alpha \mid^2 - \mid \beta \mid^2)] \sum_{n,m} [\alpha^n \alpha^{*m}/(n!m!)^{1/2}] \mid n\rangle\langle m \mid$$

Substituting this in (58) and noting that the integral gives nonzero value only when $n = m$, we obtain

$$\bar{\rho}_1 = \sum_{k=0}^{\infty} p_{1k} \mid k\rangle\langle k \mid$$

where p_{1k} is the coefficient of the Laguerre distribution[25]

$$\begin{aligned} p_{1k} &= (1 - v_0)\, v_0{}^k e^{-N_s/(N+1)} L_k[-N_s/N(N+1)] \\ v_0 &= N/(N+1) \end{aligned} \tag{59}$$

Thus

$$\begin{aligned} \pi_{s1} &= 2 \int_0^{\infty} e^{-\rho_0 y} \rho_1 e^{-\rho_0 y}\, dy \\ &= 2 \sum_{k=0}^{\infty} p_{1k} \int_0^{\infty} e^{-\rho_0 y} \mid k\rangle\langle k \mid e^{-\rho_0 y}\, dy \\ &= (1 - v_0)^{-1} \sum_{k=0}^{\infty} v_0^{-k} p_{1k} \mid k\rangle\langle k \mid \end{aligned}$$

Therefore, the operator π_s that maximizes the signal-to-noise ratio is

$$\pi_s = \sum_{k=0}^{\infty} \{e^{-N_s/(N+1)} L_k[-N_s/N(N+1)] - 1\} \mid k\rangle\langle k \mid \tag{60}$$

When signal is not present, $p_{0k} = (1 - v_0)\, v_0{}^k$, and we see that $e^{-N_s/(N+1)} L_k[-N_s/N(N+1)]$ is in fact the likelihood ratio p_{1k}/p_{0k}. Thus, π_s in (50) is analogous to the statistic

$$f(x) = [p_1(x;\, \theta)/p_0(x)] - 1$$

that maximizes the signal-to-noise ratio in the classical theory.[26]

We decide that a signal is present whenever the likelihood ratio $e^{-N'/(N+1)} L_k[-N_s/N(N+1)]$ exceeds a decision level λ. But this is precisely the strategy obtained by Helstrom for the Neyman–Pearson criterion (Helstrom,[2] p. 270). Thus, π_s and the Neyman–Pearson detector are equivalent in performance, and Helstrom has given the performance curves.[22] The maximum signal-to-noise ratio for such a detector is given by

$$\begin{aligned} D_r{}^2 &= \operatorname{Tr} \bar{\rho}_1 \pi_s = e^{-N_s/(N+1)} \sum_{k=0}^{\infty} L_k[-N_s/N(N+1)]\langle k \mid \bar{\rho}_1 \mid k\rangle - 1 \\ &= (1 - v_0)\, e^{-2N_s/(N+1)} \sum_{k=0}^{\infty} \{L_k[-N_s/N(N+1)]^2\} - 1 \end{aligned}$$

The sum above can be evaluated by a generating function formula of the generalized Laguerre polynomial (Erdelyi *et al.*,[27] p. 189, Eq, (20)]. The result is

$$D_r{}^2 = I_0\{2N_s/[N(N+1)]^{1/2}\} - 1 \tag{61}$$

For large N, $D_r^2 = N_s^2/N^2$. The ratio D_r^2 approaches infinity as $N \to 0$. This is not surprising, for π_s of (60) becomes unbounded as $N \to 0$.

5.2. A Coherent Signal of Known Phase—Small Signal

When the phase ψ of the signal is known, the problem of finding an operator that maximizes D^2 is more difficult. Therefore, we consider the case of small signal first. We need an expansion of

$$\rho_1 = (1 - v_0) \exp[-w(a - \mu)^+ (a - \mu)]$$

for small-signal parameter μ. If we ignore the factor $(1 - v_0)$ for the moment, $\{\exp[-w(a - \mu)^+ (a - \mu)]\}$ with $w \in (0, \infty)$ is a semigroup. We put

$$\begin{aligned} V(w) &= \exp[-w(a - \mu)^+ (a - \mu)] = \exp[-w(a^+a - \mu^*a - \mu a^+ + |\mu|^2)] \\ &= \exp[-w(\mathcal{N} + A)] \end{aligned}$$

where $\mathcal{N} = a^+a$ and $A = -(\mu^*a + \mu a^+) + |\mu|^2 I$ is the perturbing operator of semigroup $U(w) = e^{-w\mathcal{N}}$. The three terms of A are of the same order of magnitude;

$$\operatorname{Tr} \rho_1\mu^*a = \operatorname{Tr} \mu\rho_1a^+ = \operatorname{Tr} \rho_1 |\mu|^2 = |\mu|^2$$

and none can be ignored.

Since $V(w)$ is a semigroup, we have the following successive approximation of $V(w)$[28]:

$$V(w) = \sum_{n=0}^{\infty} U_n(w)$$

$$U_{n+1}(w) = -\int_0^w U(w - s)\, A U_n(s)\, ds$$

where $U_0(w) = U(w)$. The first-order term is

$$\begin{aligned} U_1(w) &= -\int_0^w e^{-(w-s)\mathcal{N}} A e^{-s\mathcal{N}}\, ds \\ &= -|\mu|^2 w e^{-w\mathcal{N}} + (1 - v_0)(\mu a^+ e^{-w\mathcal{N}} + e^{-w\mathcal{N}} \mu^* a) \end{aligned}$$

where we have used the operational rules (Louisell,[16] p. 111)

$$e^{-w\mathcal{N}} a = a e^{w} e^{-w\mathcal{N}}, \qquad e^{-w\mathcal{N}} a^+ = e^{-w} a^+ e^{-w\mathcal{N}} \tag{62}$$

Therefore, ρ_1 is approximately

$$\rho_1 \simeq \rho_0 - |\mu|^2 w\rho_0 + (1 - v_0)(\mu a^+ \rho_0 + \mu^* \rho_0 a)$$

π_s based on the above approximation is

$$\pi_s = 2(1 - v_0) \int_0^\infty e^{-\rho_0 y}(\mu a^+ \rho_0 + \mu^* \rho_0 a)\, e^{-\rho_0 y}\, dy - |\mu|^2 wI$$
$$= [2/(2N + 1)](\mu^* a + \mu a^+) - |\mu|^2 wI \tag{63}$$

where we used the following identities derivable from (62):

$$[\exp(-\rho_0 y)]\, a = a \exp(-e^w \rho_0 y), \qquad a^+ \exp(-\rho_0 y) = [\exp(-e^w \rho_0 y)]\, a^+$$

The first term in (63) is identical to the threshold detection operator π_t of Helstrom's (Helstrom,[4] p. 166), who obtained it by solving the equation

$$2(\partial \rho_1 / \partial \mid \mu \mid)|_{|\mu|=0} = \rho_0 \pi_t + \pi_t \rho_0$$

Since π_s of (63) does not satisfy (54), its signal-to-noise ratio D_s^2 must be computed from (52). The result is

$$D_s^2 = 4N_s/(2N + 1) \tag{64}$$

This agrees with the signal-to-noise ratio of the threshold operator given by Helstrom. As shown at the end of Section 4, as N becomes large, i.e., in the classical limit, $D_s^2 = 2E/kT$, where $E = N_s \hbar\omega$ is the energy in the signal. Unlike the signal-to-noise ratio D_r^2 of (61), D_s^2 remains bounded as $N \to 0$.

5.3. Coherent Signal of Known Phase—Arbitrary Signal

Finally, we seek the operator that maximizes D^2 for a coherent signal of known phase at an arbitrary signal level. As before, we wish to compute

$$\pi_{s1} = \int_0^\infty e^{-\rho_0 y} 2\rho_1 e^{-\rho_0 y}\, dy$$

But it is apparently not possible to express π_{s1} in a closed form. We can, however, compute the matrix element $\langle n \mid \pi_{s1} \mid m \rangle$:

$$\langle n \mid \pi_{s1} \mid m \rangle = 2 \int_0^\infty \langle n \mid e^{-\rho_0 y} \rho_1 e^{-\rho_0 y} \mid m \rangle\, dy$$
$$= 2(1 - v_0)^{-1}(v_0^{\,n} + v_0^{\,m})^{-1} \langle n \mid \rho_1 \mid m \rangle$$

where we have used the relation

$$(\exp -\rho_0 y) \mid n \rangle = \{\exp[-(1 - v_0)\, v_0^{\,n} y]\} \mid n \rangle$$

The matrix element $\langle n \mid \rho_1 \mid m \rangle$ was computed in (28). Therefore,

$$\langle n \mid \pi_s \mid m \rangle = [2 v_0^{\,n} e^{-N_s/(N+1)} / (v_0^{\,n} + v_0^{\,m})](m!/n!)^{1/2} (\mu/N)^{n-m}$$
$$\times L_m^{(n-m)}[-N_s/N(N + 1)] - \delta_{nm} \qquad (n \geqslant m) \tag{65}$$

The corresponding signal-to-noise ratio is

$$D_c^2 = \mathrm{Tr}\, \rho_1 \pi_s = \sum_{n,m} \langle n \mid \pi_s \mid m\rangle\langle m \mid \rho_1 \mid n\rangle - 1$$

$$= 2(1 - v_0)^{-1} \sum_{n,m} (v_0^n + v_0^m)^{-1} |\langle n \mid \rho_1 \mid m\rangle|^2 - 1$$

$$= (1 - v_0)^{-1} \left\{ \sum_{n=0}^{\infty} v_0^n |\langle n \mid \rho_1 \mid n\rangle|^2 \right.$$

$$\left. + 4 \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} (v_0^{m+k} + v_0^m)^{-1} |\langle m + k \mid \rho_1 \mid m\rangle|^2 \right\} - 1$$

where the first term represents the sum of matrix elements along the diagonal and the second is the sum over the elements below the diagonal. Substituting for $\langle n \mid \rho_1 \mid m\rangle$ and using a generating function for the Laguerre polynomials to evaluate the sums, we obtain

$$D_c^2 = 2 \sum_{k=0}^{\infty} \frac{v_0^{k/2} \epsilon_k}{1 + v_0^k} I_k \left\{ \frac{2N_s}{[N(N + 1)]^{1/2}} \right\} - 1 \tag{66}$$

where ϵ_k is the Neumann number; $\epsilon_0 = 1$; and $\epsilon_k = 2$ for all $k \geqslant 1$.

The difference between the signal-to-noise ratio D_c^2 of a completely known signal and the signal-to-noise ratio D_r^2 of a signal with random phase is nonnegative:

$$D_c^2 - D_r^2 = 4 \sum_{k=1}^{\infty} \frac{v_0^{k/2}}{1 + v_0^k} I_k \left\{ \frac{2N_s}{[N(N + 1)]^{1/2}} \right\} \geqslant 0$$

For a very small signal, D_c^2 of (66) reduces to $4N_s/(2N + 1)$, which is the signal-to-noise ratio of the threshold detector. However, there are terms of order N_s^2 with N^{-1} dependence that increases D_c^2 without bound as $N \to 0$. In fact, the matrix element $\langle n \mid \pi_s \mid m\rangle$ appears to be unbounded as $N \to 0$. We consider this interesting question in more detail next.

5.4. Coherent Signal of Known Phase at Zero Thermal Radiation

When $v_0 = 0$, we recall that the density operators ρ_1 and ρ_0 become, respectively, $\rho_1 = \mid \mu\rangle\langle \mu \mid$ and $\rho_0 = \mid 0\rangle\langle 0 \mid$. Therefore,

$$\pi_{s1} = 2 \int_0^{\infty} [\exp(- \mid 0\rangle\langle 0 \mid y)] \mid \mu\rangle\langle \mu \mid [\exp(- \mid 0\rangle\langle 0 \mid y)]\, dy$$

Now consider $\pi_{s1} \mid f\rangle$, where $\mid f\rangle$ is in $\mathscr{H}$. Expanding the exponent and noting that $\langle 0 \mid 0\rangle = 1$ and $\langle 0 \mid \mu\rangle = \exp(-\mid \mu \mid^2/2)$, we have

$$[\exp(-\mid 0\rangle\langle 0 \mid y)] \mid \mu\rangle = \mid \mu\rangle + [\exp(-\mid \mu \mid^2/2)](e^{-y} - 1) \mid 0\rangle$$

Letting $\langle \mu \mid f \rangle = f(\mu^*) \exp(-|\mu|^2/2)$, we obtain

$$\pi_{s1} \mid f \rangle = 2[\exp(-|\mu|^2/2)] \int_0^\infty [f(\mu^*) + (e^{-y} - 1) f(0)] \times \{| \mu \rangle + [\exp(-|\mu|^2/2)](e^{-y} - 1) \mid 0 \rangle\} \, dy$$

π_{s1} is bounded on $\mid 0 \rangle$ and on $\mid f \rangle$ such that both $f(0)$ and $f(\mu^*)$ vanish. But π_{s1} is unbounded everywhere else. Clearly, the present theory does not yield a useful operator: π_{s1} is neither bounded everywhere nor does it have its domain dense in $\mathscr{H}$.

A similar situation exists in the detection of the direction of spin in an example introduced by Helstrom.[2] We receive a beam of spin-$\frac{1}{2}$ particles along the y axis, and the particles have spin either in the z axis (hypothesis H_0) or in the x axis (hypothesis H_1). We are to decide between two density operators:

$$\rho_0 = \tfrac{1}{2}(I + \sigma_z), \qquad \rho_1 = \tfrac{1}{2}(I + \sigma_x)$$

where σ_x and σ_z are the Pauli spin matrices (Dirac,[15] p. 149)

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and I is the two by two identity matrix.

To compute π_{s1}, we note that the eigenvalues of ρ_0 are 0 and 1 and the representation of $e^{-\rho_0 y}$ on the basis where σ_z is diagonal is (Friedman,[29] p. 121)

$$e^{-\rho_0 y} = \begin{bmatrix} e^{-y} & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore,

$$\pi_{s1} = \int_0^\infty \begin{bmatrix} e^{-zy} & e^{-y} \\ e^{-y} & 1 \end{bmatrix} dy$$

and we see that $\pi_s\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ is bounded but $\pi_s\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ is not.

The fact that the present theory does not yield a useful operator in these examples shows that the definition of the signal-to-noise ratio D^2 must be reconsidered. We recall that D^2 was defined from analogy with the signal-to-noise ratio in the classical theory. We note also that the theory encountered difficulty only in physical situations that have no classical analog. Apparently the definition of D^2, like some concepts of classical mechanics, does not carry over into strictly quantum mechanical phenomena.

6. SUMMARY

The problem of detecting a completely known coherent signal of arbitrary frequency in the background of thermal radiation was considered. In particular, the detectability limit at optical frequencies was sought, since the corresponding result for low frequencies is well known. Although a closed-form expression for the detectability limit was not obtained, an algorithm was derived that enabled the calculation of the limit to any accuracy. The method is particularly applicable to optical frequencies, where the noise due to thermal radiation is small. The algorithm was shown to converge to the known result in the limit of zero thermal radiation. Some curves were generated showing the detectability limit. They represent the best performance possible without violating the laws of physics and as such are useful as a basis for comparing various practical communication systems at optical frequencies.

In addition, the form of an operator that maximizes a signal-to-noise ratio was specified. The signal-to-noise ratio was defined in analogy with the classical theory and it led to useful results whenever the physical situation had a classical analog. In particular, for a coherent signal of random phase, the operator that maximized the signal-to-noise ratio was identical to the one obtained by applying the Neyman–Pearson criterion. However, in situations that had no classical analog, an operator that maximized the signal-to-noise ratio did not exist in the usual sense of quantum mechanics.

ACKNOWLEDGMENT

The author wishes to thank Professor A. V. Balakrishnan of the University of California, Los Angeles, for his guidance and encouragement throughout the period of this research. The author also wishes to thank Professor C. W. Helstrom of the University of California, San Diego, for his helpful suggestions.

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